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Optimal Liquidation Strategy of Multi-assets Based on Minimum Loss Probability

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Abstract. Based on the minimum loss probability criterion, this paper discusses the optimal strategy in multi-asset liquidation. First, we give the framework of the multi-asset liquidation problem and obtain the boundary conditions of the optimal liquidation strategy under the assumption of linear price impact functions and transform the multi-asset liquidation problem into the portfolio liquidation problem. On this basis, the asymptotic solution and numerical solution of the optimal liquidation strategy are obtained. Then, we simulate the trajectories of the optimal liquidation strategy and analyze the effects of parameters changes.

Keywords: Minimum loss probability, multi-asset liquidation, permanent impact, temporary impact, optimal liquidation strategy

1. Introduction

With the rapid development of computer and Internet technology, great changes have taken place in the transaction modes in the modern financial market. Algorithmic trading characterized by high-frequency data processing and automatic computer ordering emerges. Algorithmic trading refers to the general term of making trading decisions, submitting orders and managing orders by computer. Brunnermeier and Pedersen (2008) and Scholtus et al. (2014) believe that algorithmic trading has significant effects on improving market efficiency, increasing liquidity and reducing transaction costs of large positions.

However, improper use of algorithmic trading may lead to large consumption of market liquidity and a substantial increase in market volatility in a short period, bringing certain risks to the market and even triggering a huge crisis. The Flash Crash in the US stock market in 2010 and the fat finger accident of Everbright Securities in China in 2013 have proved this point, which causes the stock price to fluctuate greatly in a short period. Therefore, the in-depth study of algorithm trading is particularly essential.

One of the core concerns of algorithmic trading is the short-term execution of the large positions of investors in financial markets. Due to the limited market liquidity, the immediate trading of large positions will inevitably impact the prices of risky assets, thus increasing transaction costs. Therefore, investors need to split their positions into small orders that can be executed in batches. However, such the split order transaction will prolong the transaction time and thus increase the risk due to uncertain volatility. Therefore, a good liquidation strategy must balance the transaction costs and risks.

Many models have been developed in the studies of market price impact and optimal execution strategies. The first-class models include the mean-variance models (Almgren and Chriss 1999 2000), which assume that the asset price process obeys the arithmetic Brownian motion. The price impact in the trading process can be divided into two parts: the permanent price impact and the temporary price impact. The optimal solution to this kind of model seeks a balance between minimizing the price impact and minimizing the time risk. The cost functions include the following three kinds. The first minimizes the risk criterion, such as using the variance (Engle and Ferstenberg 2007, Forsyth et al. 2012), the quadratic variation (Almgren and Chriss 2000) or the value at risk (VaR) (Gatheral and Schied 2013) as the risk measurement to obtain the optimal strategy. The second is the utility function criterion, such as using the power function or exponential function (He and Mamaysky 2005, Schied and Schoneborn 2009) as the utility function and maximizing it to obtain the optimal execution strategy. The third minimizes the expectation of execution cost. This criterion minimizes the expected cost by constructing a model to characterize the dynamic properties of the bid orders and the ask orders (Bertsimas and Lo 1998, Alfonsi et al. 2010, Obizhaeva and Wang 2013).

The second-class models based on the Bertsimas-Lo model Bertsimas and Lo (1998) assume that the asset price process obeys the geometric Brownian motion and the optimal liquidation strategies need to be solved by dynamic programming method. Forsyth et al. (2012) established the HJB equation of the optimal execution problem under the expected returns and quadratic variation criteria and analyzed the optimal liquidation strategy of a portfolio based on the HJB equation.

The third-class models based on the limited order book (LOB) model assume that the price impact produced by transactions has attenuating effect on the price. That is, the price impact can be regarded as a nonlinear time-varying function of the transaction volume. The permanent price impact and temporary price impact are two special cases of the function. The optimal execution problem under this assumption has been widely discussed (Gatheral 2010, Obizhaeva and Wang 2013, Alfonsi et al. 2012, Gatheral et al. 2012).

In a recent paper, Jin (2017) proposed the loss probability (the probability that the actual liquidation cost is higher than a given value) utility function based on the transaction cost model that was proposed by Almgren and Chriss (2000). The author derived the asymptotic solution for the liquidation model without a time constraint and gave the numerical solution considering a time constraint. The inadequacy of this study is that the article only considered the optimal liquidation of a single asset.

However, in a real trading environment, institutional investors such as mutual funds rarely hold only one risky asset for trading, but usually hold multiple risky assets or several portfolios. To earn profits or avoid risks, they need to liquidate them within a specified period. Different from transactions in the case of a single asset, due to the possible correlation between multiple assets, making a strategy for each asset alone may not achieve the desired liquidation effect. Therefore, this correlation has to be taken into account when formulating liquidation strategies for multiple assets. This paper establishes the correlation between assets by setting the Brownian motion process and the market impact process of assets and extends the Jin (2017) model to a multidimensional model to consider the optimal liquidation of a portfolio.

The rest of this paper is as follows. The second part establishes a multi-asset framework and discusses the properties of the asymptotic solution and numerical solution of a portfolio.

The third part gives the simulation results of a portfolio liquidation problem. The last part is the conclusion.

2. Optimal Liquidation Strategy for Multi-asset

2.1 Multi-asset Liquidation Cost Model

Assuming that an institutional investor holds M kinds of risky assets, the initial positions of the assets can be expressed as the vector$x_0 = \left(x_0^1, x_0^2, \ldots, x_0^M\right)'$, and the initial price vector is $S_0 = \left(S_0^1, S_0^2, \ldots, S_0^M\right)'$. The investor needs to liquidate all assets within $T = N\tau$, where τ is the unit time. At time $k\tau$, we define the positions of the multiple assets as $x_k = \left(x_k^1, x_k^2, \ldots, x_k^M\right)'$, where $x_{k'}^j, 1 \le k \le N, 1 \le j \le M$, denotes the amount of remaining position of the jth asset, and the boundary condition is $x_N = 0$. During the period $(k\tau, (k+1)\tau)$, the liquidation strategy of the asset is $n_k = \left(n_k^1, n_k^2, \ldots, n_k^M\right)'$, where $n_k^j, 1 \le k \le N, 1 \le j \le M$, denotes the amount of the jth asset liquidated during this period. Obviously, there is a recursive relationship between x_k and n_k as follows:

$$n_k = x_{k-1} - x_k, 1 \le k \le N$$

It is generally assumed that the basic price process S_t of M-dimensional assets obeys the M-dimensional arithmetic Brownian motion for convenience of the analysis. We assume that the permanent price impact function is linear in terms of the number of shares that are traded. The permanent price impact of a single stock not only has a lasting influence on its future price but also on the prices of the other stocks in the portfolio. Furthermore, we assume that the temporary price impact function has a linear relationship with the liquidation quantity of a single stock in a period. Besides, the temporary impact only affects the current transaction of the stock and does not affect the other stocks in the portfolio. Therefore, at time $k\tau$, the asset price vector S_k can be expressed as

$$S_k = S_{k-1} + \sigma\tau^{\frac{1}{2}}\xi_k - g\left(n_k\right) \tag{1}$$

where σ represents the volatility matrix, which is defined as a lower triangular matrix. In addition, the covariance matrix of M-dimensional assets is $\Sigma = \sigma\sigma'$, which is a positive definite matrix. ξ_k is a M-dimensional random vector that is composed of M independent and identically distributed standard normal random variables, and $g(\cdot)$ is the permanent impact function.

We define $h(\cdot)$ as the temporary price impact function and the actual execution price of the portfolio in period $(k\tau, (k+1)\tau)$ can be expressed as

$$\tilde{S}_k = S_{k-1} - h\left(n_k\right) \tag{2}$$

Then, we substitute equation (1) into equation (2) and deduce

$$\tilde{S}_k = S_0 + \sum_{j=1}^{k-1} \sigma\tau^{\frac{1}{2}}\xi_j - \sum_{j=1}^{k} h\left(n_k\right) - g\left(n_k\right) \tag{3}$$

To facilitate the analysis of this problem, we further assume that

$$g\left(n_k\right) = \lambda n_k$$
$$h\left(n_k\right) = v n_k$$

where $\lambda = \left(\lambda^{ij}\right) \in \mathbb{R}^{M*M}$ is the permanent price impact coefficient matrix, $v = \operatorname{diag}\left(v^1, v^2, \cdots, v^M\right)$ is the temporary price impact coefficient matrix.

We define the liquidation cost of the portfolio as C, which can be expressed as

$$C = x_0' S_0 - \sum_{k=1}^{N} n_k' \tilde{S}_k$$

Therefore, we can give the expression of the liquidation cost and its expectation and variance.

Lemma 1 *The expression of the portfolio liquidation cost C is*

$$C = -\tau^{\frac{1}{2}} \sum_{k=1}^{N} x_k' \sigma \xi_k + \frac{1}{2} x_0' \lambda x_0 + \sum_{k=1}^{N} (x_{k-1} - x_k)' \left(v - \frac{1}{2}\lambda\right) (x_{k-1} - x_k) \tag{4}$$

Proof.

$$\begin{aligned} C &= x_0' S_0 - \sum_{k=1}^{N} n_k' \tilde{S}_k \\ &= x_0' S_0 - \sum_{k=1}^{N} n_k' S_0 - \sum_{k=1}^{N} n_k' \sum_{i=1}^{k-1} \sigma \tau^{\frac{1}{2}} \xi_i \\ &\quad - \sum_{k=1}^{N} n_k' \lambda \sum_{i=1}^{k} n_i + \sum_{k=1}^{N} n_k' v n_k \\ &= - \sum_{k=1}^{N} x_k' \sigma \tau^{\frac{1}{2}} \xi_k - \sum_{k=1}^{N} (x_{k-1} - x_k)' \lambda \sum_{i=1}^{k} n_i \\ &\quad + \sum_{k=1}^{N} n_k' v n_k \\ &= - \sum_{k=1}^{N} x_k' \sigma \tau^{\frac{1}{2}} \xi_k \\ &\quad - \sum_{k=1}^{N} (x_{k-1} - x_k)' \lambda (x_0 - x_{k-1}) \\ &\quad + \sum_{k=1}^{N} (x_{k-1} - x_k)' v (x_{k-1} - x_k) \\ &= -\tau^{\frac{1}{2}} \sum_{k=1}^{N} x_k' \sigma \xi_k + \frac{1}{2} x_0' \lambda x_0 + \\ &\quad \sum_{k=1}^{N} (x_{k-1} - x_k)' \left(v - \frac{1}{2}\lambda\right) (x_{k-1} - x_k) \end{aligned}$$

■

From the above deduction, we can see that the liquidation cost C is the function of the volatility matrix σ, the permanent price impact coefficient matrix λ and the temporary price impact coefficient matrix v. It means that the liquidation cost is affected by the correlation between the different assets.

Furthermore, the liquidation cost C is a random variable, so its expectation and variance can be easily obtained.

Lemma 2 *The expectation μ_C and variance σ_C^2 of the portfolio liquidation cost C can be respectively expressed as*

$$\mu_C = \frac{1}{2} x_0' \lambda x_0 + \sum_{k=1}^{N} (x_{k-1} - x_k)' \left(v - \frac{1}{2}\lambda\right) (x_{k-1} - x_k) \tag{5}$$

$$\sigma_C^2 = \tau \sum_{k=1}^{N} x_k' \sigma \sigma' x_k \tag{6}$$

From equation (6), we can see that the variance is the quadratic form of the execution strategy.

Proof. From equation (4), the liquidation cost C can be regarded as a function of the random variable ξ, therefore, equation (5) is established. Its variance can be revealed as

$$\begin{aligned} \sigma_C^2 &= E\left[C - \mu_C\right]^2 \\ &= E\left[- \sum_{k=1}^{N} x_k' \sigma \tau^{\frac{1}{2}} \xi_k\right]^2 \\ &= \tau \sum_{k=1}^{N} E\left[x_k' \sigma \xi_k\right]\left[x_k' \sigma \xi_k\right]' \\ &= \tau \sum_{k=1}^{N} E\left[x_k' \sigma \xi_k \xi_k' \sigma' x_k\right] \\ &= \tau \sum_{k=1}^{N} x_k' \sigma \sigma' x_k \end{aligned} \tag{7}$$

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2.2 Optimal Liquidation Strategy Model under Loss Probability Measure

Jin (2017) proposed a loss function measure as the utility function of an asset liquidation based on the Almgren and Chriss (2000) model. The optimal liquidation strategy model can be given as follows:

$$\min P\left(C_{n(N)} > b\right)$$
$$\text{s.t.} \sum_{k=1}^{N} n_k = X$$
$$n_k \geq 0, 1 \leq k \leq N$$

where N is the length of the liquidation, X is the number of assets to be liquidated, n_k is the number of positions to be liquidated during the period $(k\tau, (k+1)\tau)$, $n_k \geq 0$ represents a pure selling strategy, b is a given threshold of the liquidation cost, $C_{n(N)}$ is the liquidation cost under the liquidation strategy $n(N)$, and $P\left(C_{n(N)} > b\right)$ is the loss probability that the actual liquidation cost is higher than the given threshold. We can obtain the liquidation strategy by minimizing the loss probability.

Based on Jin (2017)'s work, we extend the optimal execution strategy model to the multi-asset case. Next, we will discuss several typical liquidation strategies, which are the uniform order strategy known as the naive execution strategy in Almgren and Chriss (2000) and elsewhere as the simple strategy, the final order strategy and the initial order strategy. In addition, we will give the cost expectations and cost variances of the corresponding strategies.

Lemma 3 *The uniform order strategy* $n(k) = \left(\frac{x_0^1}{N}, \ldots, \frac{x_0^M}{N}\right)'$ *has the minimum expectation of the liquidation cost, which is denoted as* $\mu_{C,\min}$*:*

$$\begin{aligned}
&\mu_{C,\min} \\
&= \frac{1}{2}x_0'\lambda x_0 + \\
&\quad \sum_{k=1}^{N}\left(\frac{x_0^1}{N}, \ldots, \frac{x_0^M}{N}\right)\left(v - \frac{1}{2}\lambda\right)\left(\frac{x_0^1}{N}, \ldots, \frac{x_0^M}{N}\right)' \\
&= \frac{1}{2}x_0'\lambda x_0 + \frac{1}{N}x_0'\left(v - \frac{1}{2}\lambda\right)x_0
\end{aligned}$$

Lemma 4 *If all positions are liquidated in the last period, the liquidation strategy is as follows:*

$$\begin{cases} n(N) = \left(x_0^1, x_0^2, \ldots, x_0^M\right)' \\ n(k) = (0, \ldots, 0)', \forall k \in \{1, \ldots, N-1\} \end{cases}$$

The corresponding variance of the liquidation cost is the largest, which is denoted as $\sigma^2_{C,\max}$*:*

$$\sigma^2_{C,\max} = \tau N x_0'\sigma\sigma' x_0$$

Lemma 5 *If all positions are liquidated in the first period, the liquidation strategy is as follows:*

$$\begin{cases} n(1) = \left(x_0^1, x_0^2, \ldots, x_0^M\right)' \\ n(k) = (0, \ldots, 0)', \forall k \in \{2, \ldots, N\} \end{cases}$$

The corresponding variance of the liquidation cost is the smallest, which is denoted as $\sigma^2_{C,\min}$*:*

$$\sigma^2_{C,\min} = 0 \tag{8}$$

For a given threshold b, the loss probability is defined as the probability that the liquidation cost exceeds the threshold b, which is defined as $P(C > b)$. since the liquidation cost C is a random variable with the expectation of μ_C and the variance of σ_C^2 obeying a normal distribution, the loss probability can be transformed as

$$P(C > b) = 1 - \Phi\left(\frac{b - \mu_C}{\sigma_C}\right) \tag{9}$$

where Φ is the standard normal cumulative distribution. The optimization goal of the model is to minimize the loss probability. Therefore, the optimization model can be written as

$$\min P(C > b) \tag{10}$$
$$\text{s.t.} \sum_{k=1}^{N} n_k^j = x_0^j, 1 \leq j \leq M$$
$$n_k^j \geq 0, 1 \leq k \leq N, 1 \leq j \leq M$$

Combined with equation (9), the objective of the optimization model is equivalent to maximizing $\Phi(z)$, which means to maximize

z. Therefore, the optimization model can be rewritten as follows:

$$\begin{aligned} &\min -z, \\ &\text{s.t. } z\sigma_C^2 + \mu_C - b \le 0 \\ &-x \le 0 \\ &-n_k^j \le 0, 1 \le k \le N, 1 \le j \le M \\ &\sum_{k=1}^{N} n_k^j = x_0^j, 1 \le j \le M \end{aligned}$$

2.3 Optimal Liquidation Strategy Without Time Constraints

In this part, we will discuss the optimal liquidation strategy without time constraints. We record the optimal liquidation strategy as $n(*)$ and the corresponding minimum loss probability as P_b^*, which can be expressed as

$$n(*) = \arg \min_{n=(n_1,n_2,\cdots,n_N)} P(C > b)$$

$$P_b^* = \min_{n=(n_1,n_2,\cdots,n_N)} P(C > b) = P\left(C_{n(*)} > b\right)$$

From Lemma 1 we assume that the last item $\sum_{k=1}^{N}(x_k - x_{k+1})'\left(v - \frac{1}{2}\lambda\right)(x_k - x_{k+1})$ of the liquidation cost in equation (4) is greater than 0, that is, $\sum_{k=1}^{N}(x_k - x_{k+1})'\left(v - \frac{1}{2}\lambda\right)(x_k - x_{k+1}) > 0$, which also implies that the matrix $\left(v - \frac{1}{2}\lambda\right)$ is positive definite. Conversely, when $\sum_{k=1}^{N}(x_k - x_{k+1})'\left(v - \frac{1}{2}\lambda\right)(x_k - x_{k+1}) < 0$, in order to minimize the loss probability, the optimal liquidation strategy is to liquidate all assets in the first period. When $\sum_{k=1}^{N}(x_k - x_{k+1})'\left(v - \frac{1}{2}\lambda\right)(x_k - x_{k+1}) = 0$, the expectation of the liquidation cost is not related to the choice of the liquidation strategy, and thus $\mu_C = \frac{1}{2}x_0'\lambda x_0$.

With $\sum_{k=1}^{N}(x_k - x_{k+1})'\left(v - \frac{1}{2}\lambda\right)(x_k - x_{k+1}) > 0$, according to threshold b, the optimal liquidation strategy is chosen as follows. When $b \ge \frac{1}{2}x_0'\lambda x_0$, the optimal strategy corresponds to Lemma 5 in which the variance of the cost is 0 and the loss probability is 0. When $b < \frac{1}{2}x_0'\lambda x_0$ the optimal strategy corresponds to Lemma 4 in which the liquidation cost variance reaches the maximum and the loss probability will be greater than 0.5.

Using the Lagrange multiplier and Karush-Kuhn-Tucker (KKT) conditions, the boundary conditions that are satisfied by the liquidation strategy of multidimensional assets can be given by Theorem 1.

Theorem 1 *The boundary condition of optimization problem (10) is*

$$\begin{aligned} &\frac{\sum_{j=1}^{M}\left(2v^j - \sum_{l=1}^{M}\lambda^{jl}\right)\left(n_{k+1}^j - n_k^j\right)}{\sum_{j=1}^{M}\sigma^{jj^2}x_k^j} \\ &= -\frac{b-\mu_c}{\sum_{k=1}^{N}x_k'\sigma\sigma' x_k} \end{aligned} \tag{11}$$

Proof. By using the Lagrange multiplier $\theta^j (1 \le j \le M)$ and $\theta_i^j (1 \le j \le M$ and $1 \le i \le N)$, the optimal liquidation strategy n(*) needs to maximize the following equation:

$$\begin{aligned} &P(C < b) + \sum_{j=1}^{M}\theta^j\left(n_1^j + \cdots + n_N^j - x_0^j\right) \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{M}\theta_i^j n_i^j \end{aligned}$$

where $P(C < b)$ is

$$P(C < b) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{\xi^2}{2}}d\xi$$

The upper limit of the integral is $z = \frac{b-\mu_C}{\sigma_C}$. Using the Leibniz formula calculates $\frac{\partial P}{\partial n_i^j}$ as follows:

$$\begin{aligned} \frac{\partial P}{\partial n_i^j} &= \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\frac{\partial z}{\partial n_i^j} \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\frac{\sqrt{V}\frac{\partial \mu}{\partial n_i^j} - \frac{1}{2}Z\frac{\partial V}{\partial n_i^j}}{V} \end{aligned}$$

where $V = \sigma_C^2$. In addition, by setting $\mu = \mu_C$, we can calculate $\frac{\partial \mu}{\partial n_i^j}$ and $\frac{\partial V}{\partial n_i^j}$ as

$$\mu_C = \frac{1}{2}x_0'\lambda x_0 + \sum_{k=1}^{N}(x_k - x_{k+1})'\left(v - \frac{1}{2}\lambda\right)(x_k - x_{k+1})$$

$$\mu = \sum_{k=1}^{N}\sum_{j=1}^{M} x_k^j \sum_{l=1}^{M}\lambda^{jl}n_k^j + \sum_{k=1}^{N} n_k' v n_k$$

By setting $F = \sum_{k=1}^{N}\sum_{j=1}^{M} x_k^j \sum_{l=1}^{M}\lambda^{jl}n_k^j$, we can obtain

$$\frac{\partial F}{\partial n_i^j} = \sum_{k=1}^{N}\sum_{j=1}^{M}\frac{\partial x_k^j}{\partial n_i^j}\sum_{l=1}^{M}\lambda^{jl}n_k^j + \sum_{k=1}^{N}\sum_{j=1}^{M} x_k^j \frac{\partial \sum_{l=1}^{M}\lambda^{jl}n_k^j}{\partial n_i^j}$$
$$= \sum_{j=1}^{M}\sum_{k=1}^{i-1}\sum_{l=1}^{M}\lambda^{jl}n_k^j + \sum_{j=1}^{M}\sum_{k=i+1}^{N}\sum_{l=1}^{M}\lambda^{jl}n_k^j$$

Then, by setting $H = \sum_{k=1}^{N} n_k' v n_k = \sum_{j=1}^{M}\sum_{k=1}^{N} v^j n_k^{j^2}$, it can be given by

$$\frac{\partial H}{\partial n_i^j} = 2\sum_{j=1}^{M} v^j n_i^j$$

Therefore,

$$\frac{\partial \mu}{\partial n_i^j} = \sum_{j=1}^{M}\sum_{k=1}^{i-1}\sum_{l=1}^{M}\lambda^{jl}n_k^j + \sum_{j=1}^{M}\sum_{k=i+1}^{N}\sum_{l=1}^{M}\lambda^{jl}n_k^j + 2\sum_{j=1}^{M} v^j n_i^j$$
$$= \sum_{j=1}^{M}\sum_{l=1}^{M}\lambda^{jl}x_0^j + \sum_{j=1}^{M}\left(2v^j - \sum_{l=1}^{M}\lambda^{jl}\right)n_i^j$$

According to $V = \tau\sum_{k=1}^{N} x_k'\sigma\sigma' x_k$, we can obtain

$$\frac{\partial V}{\partial n_i^j} = \frac{\partial V}{\partial x_k}\frac{\partial x_k}{\partial n_i^j}$$

Furthermore, we can get $\frac{\partial V}{\partial x_k} = 2\tau\sum_{k=1}^{N}\sigma\sigma' x_k$. In addition, considering $x_k = \left(\sum_{r=k+1}^{N} n_r^1, \sum_{r=k+1}^{N} n_r^2, \ldots, \sum_{r=k+1}^{N} n_r^M\right)'$, it can be deduced that

$$\frac{\partial x_k}{\partial n_i^j} = \left(\frac{\partial \sum_{r=k+1}^{N} n_r^1}{\partial n_i^j}, \cdots, \frac{\partial \sum_{r=k+1}^{N} n_r^j}{\partial n_i^j}, \cdots, \frac{\partial \sum_{r=k+1}^{N} n_r^M}{\partial n_i^j}\right)'$$
$$= \Bigg(\underbrace{0, \cdots, 0}_{j-1 \text{ items}}, \frac{\partial \sum_{\mathrm{r}=k+1}^{N} n_{\mathrm{r}}^j}{\partial n_i^j}, \underbrace{0, \cdots, 0}_{M-j \text{ items}}\Bigg)'$$

By substituting the former equation into $\frac{\partial V}{\partial n_i^j} = \frac{\partial V}{\partial x_k}\frac{\partial x_k}{\partial n_i^j}$, we have

$$\frac{\partial V}{\partial n_i^j} = 2\tau\sum_{k=1}^{N}\sigma\sigma' x_k\Bigg(\underbrace{0, \cdots, 0}_{j-1 \text{ items}}, \frac{\partial \sum_{r=k+1}^{N} n_r^j}{\partial n_i^j}, \underbrace{0, \cdots, 0}_{M-j \text{ items}}\Bigg)'$$
$$= 2\tau\sum_{k=1}^{i-1}\sum_{j=1}^{M}\sigma^{jj^2}x_k^j = 2\tau\sum_{k=1}^{i-1}\sum_{j=1}^{M}\sigma^{jj^2}x_k^j$$

According to the KKT condition $\frac{\partial P}{\partial n_i^j} = \frac{\partial P}{\partial n_{i+1}^j}$, we can get the following result that

$$\sqrt{V}\frac{\partial \mu}{\partial n_{i+1}^j} - \frac{1}{2}z\frac{\partial V}{\partial n_{i+1}^j} = \sqrt{V}\frac{\partial \mu}{\partial n_i^j} - \frac{1}{2}z\frac{\partial V}{\partial n_i^j}$$

where

$$\frac{\partial \mu}{\partial n_{i+1}^j} - \frac{\partial \mu}{\partial n_i^j} = \sum_{j=1}^{M}\left(2v^j - \sum_{l=1}^{M}\lambda^{jl}\right)\left(n_{i+1}^j - n_i^j\right)$$

$$\frac{\partial V}{\partial n_{i+1}^j} - \frac{\partial V}{\partial n_i^j} = 2\tau\sum_{k=1}^{i}\sum_{j=1}^{M}x_k^j\sigma^{jj} - 2\tau\sum_{k=1}^{i-1}\sum_{j=1}^{M}x_k^j\sigma^{jj}$$
$$= 2\tau\sum_{j=1}^{M}x_i^j\sigma^{jj}$$

With $\mu = \mu_c$, the following equation can be obtained.

$$\frac{\sum_{j=1}^{M}\left(2v^j-\sum_{l=1}^{M}\lambda^{jl}\right)\left(n_{k+1}^j-n_k^j\right)}{\sum_{j=1}^{M}\sigma^{jj^2}x_k^j} = -\frac{b-\mu_c}{\sum_{k=1}^{N}x_k'\sigma\sigma'x_k}$$

■

Theorem 1 can be further discussed as follows.

By setting $\omega^j=2v^j-\sum_{l=1}^{M}\lambda^{jl}$, equation (11) can be given as

$$\frac{\sum_{j=1}^{M}\omega^j\left(n_{k+1}^j-n_k^j\right)}{\sum_{j=1}^{M}\sigma^{jj^2}x_k^j} = -\frac{b-\mu_c}{\sum_{k=1}^{N}x_k'\sigma\sigma'x_k} \tag{12}$$

When a portfolio consists of one asset, with equation (12), we can get the following:

$$\frac{n_{k+1}-n_k}{x_k}=\frac{b-\mu_c}{(2v-\lambda)\sum_{k=1}^{N}x_k^2}$$

The above equation is the result given by Jin(2017) and the optimal liquidation strategy for a single asset can be obtained.

For multi-asset or portfolio situations, by observing the two sides of (12), we note that the left side of the equation is the ratio of two weighted sums. One is the weighted sum of the liquidation quantity difference between the period $((k+1)\tau,(k+2)\tau)$ and the period $(k\tau,(k+1)\tau)$ of $M-$assets (the weights are $(\left\{\omega^j\right\}$, where $j=1,\cdots,M)$, and another one is the weighted sum of the remaining positions of $M-$assets at time k (the weights are $\left\{\sigma^{jj^2}\right\}, j=1,\cdots,M)$. On the right side of the equation is the ratio of the difference between the cost threshold and the expected value of the liquidation cost and the quadratic form of the remaining positions in each period of the liquidation strategy. This implies that the ratio on the left of equation (12) is constant for the optimal liquidation strategy $n(*)$. Therefore, we use a constant D to express the left side of the equation as follows:

$$D=\frac{\sum_{j=1}^{M}\omega^j\left(n_{k+1}^j-n_k^j\right)}{\sum_{j=1}^{M}\sigma^{jj^2}x_k^j} \tag{13}$$

With $\sum_{j=1}^{M}\sigma^{jj^2}x_k^j>0$, $\sum_{j=1}^{M}\omega^j n_k^j$ increases or decreases monotonously. The monotony of the total number of liquidated positions in each period depends on the cost threshold. The range of the values of n_k^j is closed and bounded, that is, $\sum_{j=1}^{M}n_k^j=x_0^j$ and $n_k^j\geq 0$. Therefore, there is an optimal liquidation path to minimize the loss probability P.

From equation (13), we have

$$\sum_{j=1}^{M}\left(2\omega^j-D\sigma^{jj^2}\right)x_k^j=\sum_{j=1}^{M}\omega^j x_{k+1}^j+\sum_{j=1}^{M}\omega^j x_{k-1}^j \tag{14}$$

Assuming that all individual assets simultaneously satisfy the marginal conditions when equation (14) is established. That is, for all j, where $j=1,\cdots,M$, the following equations hold

$$\left(2\omega^j-D\sigma^{jj^2}\right)x_k^j=\omega^j x_{k+1}^j+\omega^j x_{k-1}^j \tag{15}$$

$$\left(2-\frac{D\sigma^{jj^2}}{\omega^j}\right)x_k^j=x_{k+1}^j+x_{k-1}^j \tag{16}$$

By setting $R=2-\frac{D\sigma^{jj^2}}{\omega^j}$, we have $2\omega^j-D\sigma^{jj^2}=R\omega_j$, which means that there is a linear relationship between the price impact of the jth asset price and its variance σ^{jj^2} as follows:

$$\sigma^{jj^2}=\frac{2-R}{D}\left(2v^j-\sum_{l=1}^{M}\lambda^{jl}\right) \tag{17}$$

Therefore, equation (14) can be expressed as

$$R\sum_{j=1}^{M}\omega^j x_k^j=\sum_{j=1}^{M}\omega^j x_{k+1}^j+\sum_{j=1}^{M}\omega^j x_{k-1}^j \tag{18}$$

By setting $\sum_{j=1}^{M} y_k^j = \sum_{j=1}^{M} \omega_j x_k^j$, the above equation is rewritten as follows:

$$\sum_{j=1}^{M} y_{k+1}^j - R\sum_{j=1}^{M} y_k^j + \sum_{j=1}^{M} y_{k-1}^j = 0 \tag{19}$$

Equation (19) can be considered as a second-order difference equation with regard to the series $\sum_{j=1}^{N} y_k^j$, and the boundary conditions satisfy $\sum_{j=1}^{M} y_0^j = \sum_{j=1}^{M} \omega^j x_0^j$ and $\sum_{j=1}^{M} y_N^j = \sum_{j=1}^{M} \omega^j x_N^j = 0$. Thus, we can get the following results.

Theorem 2 *Suppose that* $b < \mu_{C,\min}$, *thus,* $\sum_{j=1}^{M} y_k^j$ *increases monotonously and* $D = \frac{2-R}{\sigma^{jj^2}}\omega^j > 0$. *Then, we have*

$$\sum_{j=1}^{M} \omega^j n_k^j = \sum_{j=1}^{M} \omega^j x_0^j \left(\cos\frac{(k-1)\pi}{2N} - \cos\frac{k\pi}{2N}\right) \tag{20}$$

Proof. We have the following:

$$\sum_{j=1}^{M} y_{k+1}^j - R\sum_{j=1}^{M} y_k^j + \sum_{j=1}^{M} y_{k-1}^j = 0$$

Therefore, $\sum_{j=1}^{M} y_k^j$ satisfies a linear difference equation, and the boundary conditions are $\sum_{j=1}^{M} y_0^j = \sum_{j=1}^{M} \omega^j x_0^j$ and $\sum_{j=1}^{M} y_N^j = \sum_{j=1}^{M} \omega^j x_N^j = 0$. With $D > 0$, we can solve the above equation and get the following (Kelley and Peterson 2000, Jin 2017):

$$\sum_{j=1}^{M} y_k^j = \sum_{j=1}^{M} y_0^j \cos\frac{k\pi}{2N}$$

Further,

$$\sum_{j=1}^{M} y_{k-1}^j - \sum_{j=1}^{M} y_k^j = \sum_{j=1}^{M} y_0^j \left(\cos\frac{(k-1)\pi}{2N} - \cos\frac{k\pi}{2N}\right)$$

Then, we can get

$$\sum_{j=1}^{M} \omega^j n_k^j = \sum_{j=1}^{M} \omega_j x_0^j \left(\cos\frac{(k-1)\pi}{2N} - \cos\frac{k\pi}{2N}\right)$$

■

Theorem 3 *Suppose that* $b > \mu_{c,\min}$, *thus,* $\sum_{i=1}^{M} y_k^j$ *decreases monotonously and* $D = \frac{2-R}{\sigma^{jj^2}}\omega^j < 0$. *Then, we can get*

$$\sum_{j=1}^{M} \omega^j x_k^j = \frac{B^k - B^{2N-k}}{1 - B^{2N}} \sum_{j=1}^{M} \omega^j x_0^j \tag{21}$$

where $B = \frac{R-\sqrt{R^2-4}}{2}$.

Proof. $\mu_{c,\min}$ is the expected liquidation cost under the strategy $n(k) = \left(\frac{x_0^1}{N}, \ldots, \frac{x_0^M}{N}\right)'$, where $\forall k \in \{1, \ldots, N\}$. With $b > \mu_{C,\min}$, the loss probability is less than 0.5 that is, $P(C > b) < 0.5$. Under the given conditions, there is a $n(*)$ that satisfies $P\left(C_{n(*)} > b\right) \leq P(C > b) < 0.5$. With $b - \mu(*)_{c,\min} > 0$, we can get $D < 0$ as follows under the optimal liquidation strategy $n(*)$:

$$D = \frac{\sum_{j=1}^{M} \omega^j \left(n_{k+1}^j - n_k^j\right)}{\sum_{j=1}^{M} \sigma^{jj^2} x_k^j} < 0$$

By solving the difference equation, we have

$$\sum_{j=1}^{M} \omega^j x_k^j = \frac{B^k - B^{2N-k}}{1 - B^{2N}} \sum_{j=1}^{M} \omega^j x_0^j$$

■

Theorem 2 and Theorem 3 give the optimal liquidation path based on loss probability respectively.

2.4 Optimal Strategy under Constrained Conditions

According to Theorem 3, we obtain the equation between the remaining position to be liquidated and the total initial position of a portfolio at a period. Generally, the asymptotic solution form is difficult to obtain. Therefore, it is necessary to constrain the permanent and temporary price impact coefficient matrixes and the volatility matrix to obtain the asymptotic solution under the constraints.

2.4.1 Asymptotic Solutions with Constraints

With equation (17), equation (12) can be simplified to

$$\frac{D}{2-R}\frac{\sum_{j=1}^{M}\omega^j\left(n_{k+1}^j-n_k^j\right)}{\sum_{j=1}^{M}\omega_j x_k^j}=-\frac{b-\mu_c}{\sum_{k=1}^{N}x_k'\sigma\sigma' x_k} \tag{22}$$

It can be regarded as the liquidation of the portfolio with weight ω_j, where $j=1,\cdots,M$. Let $\sum_{j=1}^{M}\omega_j x_k^j=\tilde{x}_k$ and $\sum_{j=1}^{M}\omega_j n_k^j=\tilde{n}_k$, then the above equation can be represented as follows:

$$\frac{D}{2-R}\frac{\tilde{n}_{k+1}-\tilde{n}_k}{\tilde{x}_k}=-\frac{b-\mu_C}{\sum_{k=1}^{N}x_k'\sigma\sigma' x_k} \tag{23}$$

Let $D'=\frac{\tilde{n}_{k+1}-\tilde{n}_k}{\tilde{x}_k}$, thus, $R=2-D'$ and $B=\frac{2-D'-\sqrt{D'^2-4D'}}{2}$. With $D'>0$, equation (20) can be simplified to

$$\tilde{n}_k=\tilde{x}_0\left(\cos\frac{(k-1)\pi}{2N}-\cos\frac{k\pi}{2N}\right)$$

With $D'<0$, equation (21) can be simplified to

$$\tilde{x}_k=\frac{B^k-B^{2N-k}}{1-B^{2N}}\tilde{x}_0$$

On this basis, there is the following theorem.

Theorem 4 *Assume that $\omega^j>0$, where $j=1,\cdots,M$. There is an asymptotic solution $\mathrm{n}^*(\infty)$ as follows:*

$$\tilde{n}_k=B^{k-1}(1-B)\tilde{x}_0 \tag{24}$$

where $B=\frac{R-\sqrt{R^2-4}}{2}$.

Proof. With $b>\frac{1}{2}x_0'\lambda x_0$, there is L_0 for any $L\geq L_0$ that satisfies

$$b>\frac{1}{2}x_0'\lambda x_0+\frac{1}{L}x_0'\left(v-\frac{1}{2}\lambda\right)x_0$$

Therefore, for any $L\geq L_0$, there is an optimal strategy $\tilde{n}^*$ such that $\tilde{n}_k(N)=\tilde{x}_{k-1}(N)-\tilde{x}_k(N)$ and $\tilde{x}_k(N)=\frac{\tilde{B}_N^k-\tilde{B}_N^{2N-k}}{1-\tilde{B}_N^{2N}}\tilde{x}_0(N)$, where $\tilde{B}_N=\frac{2-\tilde{D}_N'-\sqrt{\tilde{D}_N^2-4\tilde{D}_N}}{2}$, where $\widetilde{D}_N'=\frac{\tilde{n}_{k+1}(N)-\tilde{n}_k(N)}{\tilde{x}_k(N)}=-\frac{\left(1-\tilde{B}_N\right)^2}{\tilde{B}_N}$. It can be seen that $\tilde{B}_N^k$ and $\widetilde{D}_N'$ are variables that are related to N. Therefore, the variance of the liquidation cost of a linear combination of assets can be written as

$$\begin{aligned}\tilde{V}[C]&=\tau\sigma_\omega^2\sum_{k=1}^{N}\tilde{x}_k^2(N)\\&=\tau\sigma_\omega^2\tilde{x}_0^2(N)\left[\frac{\left(\tilde{B}_N^2-\tilde{B}_N^{2N}\right)\left(1+\tilde{B}_N^{2N}\right)}{\left(1-\tilde{B}_N^2\right)\left(1-\tilde{B}_N^{2N}\right)^2}\right.\\&\qquad\left.-\frac{(N-1)\tilde{B}_N^{2N}}{\left(1-\tilde{B}_N^{2N}\right)^2}\right]\end{aligned}$$

Here, $\sigma_\omega^2=\omega'\sigma\sigma'\omega$, where $\omega=\left(\omega^1,\cdots,\omega^M\right)'$, representing the weights of the assets.

Since $\tilde{V}[C]$ is a monotonic increasing function of N, for a given liquidation variance $\tilde{V}[C]$, $\tilde{B}_N$ decreases as N increases. Because $\tilde{B}_N\in(0,1)$ is monotonous and bounded, then $\tilde{B}_N\to B$ when $N\to\infty$.

We prove $\lim_{N\to\infty}\tilde{B}_N^N=0$ and $\lim_{N\to\infty}N\tilde{B}_N^{2N}=0$ in the following. For a large N, the following inequalities hold:

$$\begin{aligned}\tilde{n}_k(N)&=\tilde{x}_{k-1}(N)-\tilde{x}_k(N)\\&=\frac{\left(1-\tilde{B}_N\right)\left(1+\tilde{B}_N^{2N-1}\right)}{1-\tilde{B}_N^{2N}}\tilde{x}_0(N)\\&>\frac{1}{N}\tilde{x}_0(N)\end{aligned} \tag{25}$$

$$\begin{aligned}\tilde{n}_k(N)&=\tilde{x}_{k-1}(N)\\&=\frac{\tilde{B}_N^{N-1}\left(1-\tilde{B}_N^2\right)}{1-\tilde{B}_N^{2N}}\tilde{x}_0(N)\\&<\frac{1}{N}\tilde{x}_0(N)\end{aligned} \tag{26}$$

For equation (25), we have $1-\tilde{B}_N>\frac{1}{N}\frac{1-\tilde{B}_N^{2N}}{1+\tilde{B}_N^{2N-1}}$. Then, the left side approaches $1-B$ and the right side approaches 0 as $N\to\infty$. This implies that $\tilde{B}_N\to B<1$ and $\lim_{N\to\infty}\tilde{B}_N^N=0$.

For equation (26), we have $N\tilde{B}_N^{2N}<\frac{\tilde{B}_N^{N+1}\left(1-\tilde{B}_N^{2N}\right)}{1-\tilde{B}_N^2}$. Because $N\tilde{B}_N^{2N}>0,\tilde{B}_N^{N+1}\to 0$ and $\tilde{B}_N^{2N}\to 0$ as $N\to\infty,\lim_{N\to\infty}N\tilde{B}_N^{2N}=0$.

Therefore, by letting $\tilde{x}_k = \lim_{N\to\infty} \tilde{x}_k(N)$ and $\tilde{n}_k = \lim_{N\to\infty} \tilde{n}_k(N)$, we can get

$$\tilde{x}_k = B^k \tilde{x}_0$$

$$\begin{aligned} \tilde{n}_k &= \lim_{N\to\infty} \tilde{x}_{k-1}(N) - \lim_{N\to\infty} \tilde{x}_k(N) \\ &= B^{k-1}(1-B)\tilde{x}_0 \end{aligned}$$

■

Theorem 4 shows that the portfolio liquidation strategy without time constraints can be expressed as a geometric distribution function of the liquidation position. In addition, because of the geometric distribution, this liquidation path implies no memory. That is, the liquidation path of the remaining position has nothing to do with the previous liquidation actions.

2.4.2 Numerical Solutions with Constraints

To calculate the multi-asset optimal liquidation strategy under a time constraint, we improve the methods of the one-dimensional model in Jin (2017). According to the derivation process of the asymptotic solution, the numerical solution method can be given. The process is as follows.

The threshold b satisfies

$$b > \frac{1}{2} x_0' \lambda x_0 + \frac{1}{N} x_0' \left(v - \frac{1}{2}\lambda \right) x_0$$

The numerical model is solved as follows.

Step 1: Assign an initial value for D', for example, we can let $D_0' = -0.01$.

Step 2: For $i \geq 1, B_i$ is updated by

$$B_i = \frac{2 - D_{i-1}' - \sqrt{D_{i-1}'^2 - 4D_{i-1}'}}{2}$$

Step 3: For $1 \leq k \leq N$, $\sum_{j=1}^{M} x_k^j (B_i)$ is obtained by

$$\sum_{j=1}^{M} x_k^j (B_i) = \frac{B_i^k - B_i^{2N-k}}{1 - B_i^{2N}} \sum_{j=1}^{M} x_0^j$$

Step 4: For $1 \leq k \leq N$, $\sum_{j=1}^{M} n_k^j (B_i)$ is calculated by

$$\sum_{j=1}^{M} n_k^j (B_i) = \sum_{j=1}^{M} x_{k-1}^j (B_i) - \sum_{j=1}^{M} x_k^j (B_i)$$

Step5: Calculate $\mu_c (N, B_i)$ and $\sigma_C^2 (N, B_i)$ via

$$\mu_C (N, B_i) = \frac{1}{2} x_0' \lambda x_0 + \sum_{k=1}^{N} n_k' \left(v - \frac{1}{2}\lambda \right) n_k$$

$$\sigma_C^2 (N, B_i) = \tau \sum_{k=1}^{N} x_k' \sigma \sigma' x_k$$

Step6: Calculate D' via

$$D' = \frac{\tilde{n}_{k+1} - \tilde{n}_k}{\tilde{x}_k}$$

For a specific accuracy tolerance e, if $\left| D_i' - D_{i-1}' \right| \leq e$, the process stops and the optimal strategy is obtained. Otherwise, repeat the process starting from step 2.

3. Simulation Analysis

3.1 Basic Model

Consider that an institutional investor needs to liquidate a portfolio of two assets and assume that the liquidation ratios of two assets are the same. We can obtain the optimal liquidation strategy of the portfolio by the numerical method and conduct sensitivity analysis for the different parameters.The initial parameters of the model are set as shown in Table 1.

In addition, assume that the initial position vector of two assets is $x_0 = (1000000, 500000)'$ and the volatility matrix is

$$\sigma = \begin{pmatrix} 0.2 & 0 \\ 0.15 & 0.2 \end{pmatrix}$$

The permanent price impact coefficient matrix λ and the temporary price impact coefficient matrix v need to satisfy $2v^j - \sum_{l=1}^{M} \lambda^{jl} = \omega_j$, where $\omega_j, j = 1, 2$, is the weight. We set

$$\lambda = \begin{pmatrix} 1.0\text{E}-8 & 2.0\text{E}-9 \\ 2.0\text{E}-9 & 1.0\text{E}-8 \end{pmatrix}$$

Table 1 Setting of Parameters

Variable	Index	Value
M	Number of assets	2
N	Liquidation time	5
τ	Liquidation interval	1

$$\mathrm{v} = \begin{pmatrix} 1.0\mathrm{E}-7 & 0 \\ 0 & 1.0\mathrm{E}-8 \end{pmatrix}$$

Then, we can get $\omega = \left(\omega^1, \omega^2\right)' = (2.0\mathrm{E}-7, 1.0\mathrm{E}-8)'$. The lower bound of threshold b is $3.08\mathrm{E}+06$. We set the threshold b as $2.5\mathrm{E}+7$, and the optimal liquidation strategy is numerically obtained as follows:

$$n^j = (53.66\%, 25.00\%, 11.80\%, 5.90\%, 3.64\%)x_0^j$$
$$j = 1, 2$$

Hence, we can get that the liquidation cost expectation is 5072256, the variance is 1.079E+14, and the loss probability is 0.028.

3.2 Analysis of the Impact of Liquidation Cost Threshold

The threshold of liquidation cost reflects the sensitivity of investors to price impact. The higher the threshold is, the more likely investors are to ignore the impact, indicating that investors prefer complete liquidation in a concise time at the beginning of the liquidation process. We set threshold b to change from 5.0E+06 to 5.0E+07, and the optimal liquidation proportion is calculated according to the numerical model. The results are shown in Figure 1.

In Figure 1, the x-axis is the liquidation cost threshold, and the y-axis represents the allocation, that is, the liquidation proportion of the initial position. We set the liquidation time as five units, including $n1$ to $n5$.

As seen from Figure 1, with the increase of the threshold, the liquidation proportion of the position in the first period increases continuously. When the threshold increases from 5.0E+06 to 5.0E+07, the liquidation proportion of the position in the first period rises from 24.07% to 69.00%. The liquidation proportion in the second period increases first and then decreases. When the threshold is 2.0E+07, the liquidation proportion in the second period reaches 25.25%, and when the threshold increases to 5.0E+07, the proportion drops to 21.40%. Furthermore, the liquidation proportions of the third, fourth and fifth periods decrease as the threshold increases. When the threshold increases from 5.0E+06 to 5.0E+07, the three ratios decrease by 12.64%, 15.88% and 16.52%, respectively.

Table 2 shows the loss probability, expectation and variance of the liquidation cost under the optimal strategy when the liquidation cost threshold changes. With the increase in the threshold, the loss probability and the variance decrease and the expectation increases.

3.3 Analysis of the Impact of Liquidation Time

To intuitively reflect the influence of the length of liquidation N on the optimal liquidation strategy, we set $b = 8.0E + 06$ and give the optimal liquidation paths when the length of liquidation N varies from 4 to 10. Meanwhile, we compare them with the path when the time tends to infinity. The results are shown in Figure 2.

In Figure 2, the x-axis represents the liquidation time, and the y-axis represents the liquidation proportion. Each curve corresponds to the optimal liquidation proportion and loss probability under different liquidation time lengths. It can be seen that with the in-

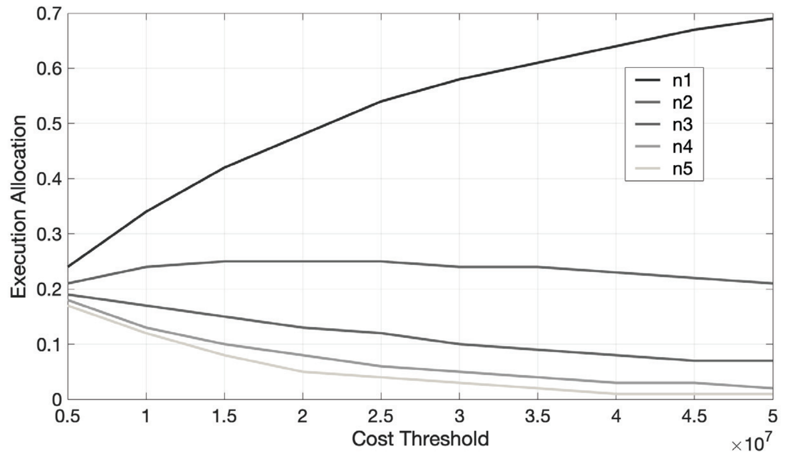

Figure 1 Relationship between the Liquidation Strategy and Liquidation Cost Threshold

Table 2 The Impact of a Threshold Change on the Liquidation Strategy

Threshold	Loss probability	Expectation	Variance
5.0E+06	0.443	3.1E+06	1.7E+14
1.0E+07	0.293	3.5E+06	1.4E+14
1.5E+07	0.164	4.0E+06	1.3E+14
2.0E+07	0.075	4.6E+06	1.2E+14
2.5E+07	0.028	5.1E+06	1.1E+14
3.0E+07	0.008	5.5E+06	1.0E+14
3.5E+07	0.002	5.9E+06	1.0E+14
4.0E+07	0.000	6.3E+06	9.7E+13
4.5E+07	0.000	6.6E+06	9.5E+13
5.0E+07	0.000	6.9E+06	9.4E+13

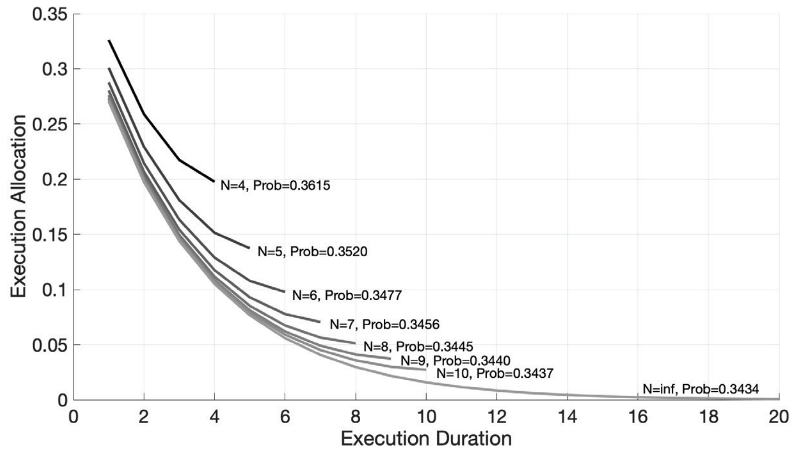

Figure 2 Influence of the Change of the Liquidation Time N on the Liquidation Strategy

crease of the liquidation time, the corresponding loss probability under the optimal strategy decreases. When the time tends to infinity, the expression of liquidation strategy is as follows:

$$n_k^j = 27.05\%(1 - 27.05\%)^{k-1} x_0^j \tag{27}$$

The expectation and variance of the liquidation cost are 2.57E+06 and 1.82E+14 respectively, and the corresponding loss probability is as low as 0.3434.

3.4 The Impact of Volatility Change

Stock volatility is an important factor affecting the liquidation strategy. When institutional investors liquidate multiple assets, different assets may be affected jointly by the same volatility components, which must be considered when formulating liquidation strategies. In the simulation, the volatility matrix is changed according to the data that are shown in Table 3 . For convenience, we set $\sigma^{11} = \sigma^{22} = z(1) * 0.2$ and $\sigma^{21} = z(2) * 0.15$. Then, we change the values of $z(1)$ and $z(2)$ and keep the other parameters unchanged. Thus, the optimal liquidation strategies can be obtained, as shown in Figure 3 and Figure 4.

The x and y axes in Figure 3 and Figure 4 show the values of $z(1)$ and $z(2)$, respectively. The z-axis represents the optimal liquidation proportion. The change of the color of the surface from the light color to the dark color means that the liquidation proportion decreases. In the first period, when $\sigma^{11} = \sigma^{22}$ is unchanged, the optimal liquidation proportion decreases as σ^{21} increases, and when σ^{21} keeps unchanged, the optimal liquidation proportion decreases as σ^{11} increases. The optimal liquidation proportions of the several remaining periods increase as the volatility increases. It means that with the increase of the volatility, the optimal liquidation strategies should tend to approach the simple strategy to balance the liquidation shares in different periods.

Then, we show the effect of volatility on loss probability through Figure 5 and Figure 6.

Figure 5 shows that the loss probability generally increases with the increase of volatility. Figure 6 shows the loss probability curves with different parameters: one sets $z(1) = 1$ and changes $z(2)$ from 0.5 to 2, and the other one sets $z(2) = 1$ and changes $z(1)$ from 0.5 to 2. The x-axis in Figure 6 shows the value of $z(1)$ or $z(2)$, and the y-axis is the loss probability. It shows that when $\sigma^{11} = \sigma^{22} = 0.2$ and σ^{12} increases from 0.075 to 0.3, the loss probability increases from 0.0148 to 0.0640, and when $\sigma^{12} = 0.15$ and $\sigma^{11} = \sigma^{22}$ increases from 0.1 to 0.4, the loss probability increases from 0.0010 to 0.1412. This result means that $\sigma^{11} = \sigma^{22}$ has a more significant effect on the loss probability than σ^{21}.

4. Conclusion

Based on the minimum loss probability criterion, this paper discusses the optimal strategy in the multi-asset liquidation problem. Our work includes three parts: the model representation, the theoretical derivation and the computer simulation.

In the model representation part, we give the framework of the multi-asset liquidation problem based on the minimum loss probability criterion and assume that the permanent price impacts of assets will affect each other while the temporary price impacts only affect the assets themselves. Under the assumptions of the linear price impact functions, we obtain the expectation and variance of the liquidation cost.

In the theoretical derivation part, we use the Lagrange multiplier method and KKT conditions to derive the boundary conditions that are satisfied by the liquidation strategy of multidimensional assets and transform the multi-asset liquidation problem into the portfolio liquidation problem. On this basis, we derive

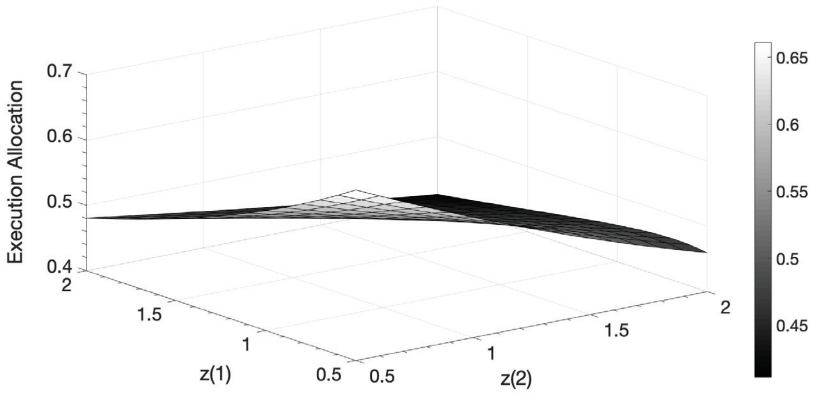

Figure 3 Optimal Liquidation Proportion with the Volatility Matrix in the 1^{st} Period

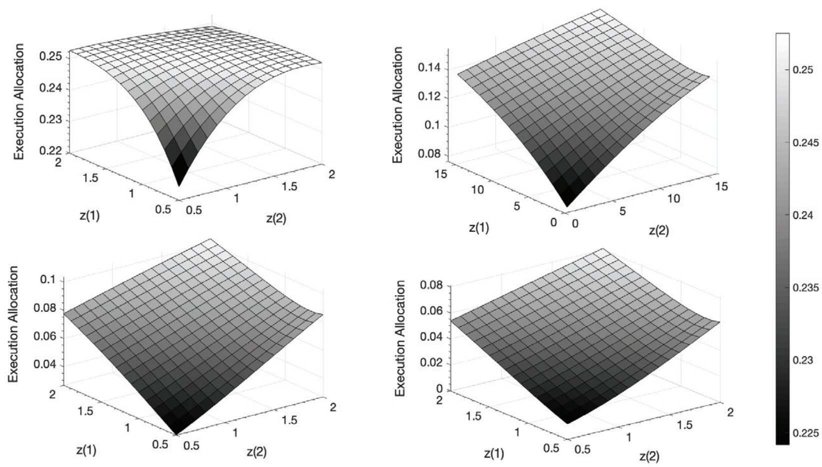

Figure 4 Optimal Liquidation Proportion with the Volatility Matrix in the $2^{\text{nd}} - 5^{\text{th}}$ Periods

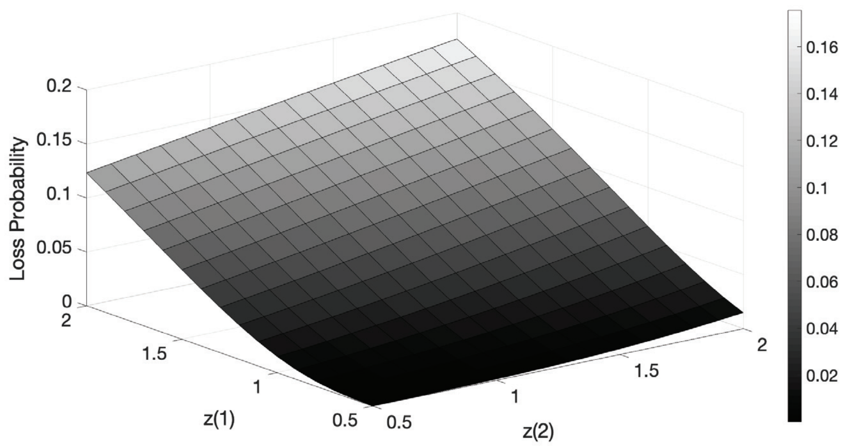

Figure 5 Loss Probability Varies with the Volatility under the Optimal Liquidation Strategy

Table 3 The Impact of a Threshold Change on the Liquidation Strategy

Variance	Initial value	Terminal value	Fluctuation
$z(1)$	0.5	2	0.1
$z(2)$	0.5	2	0.1

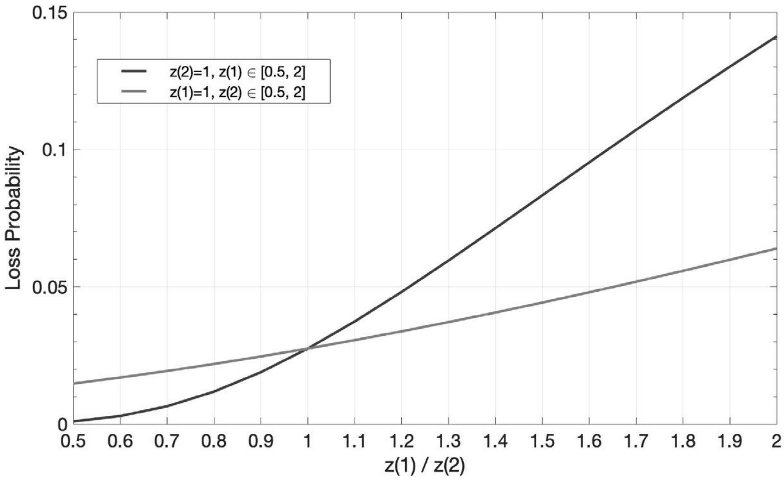

Figure 6 Minimum Loss Probability Curves with Different Volatility Matrixes

the asymptotic solution of the optimal trading strategy when the liquidation time tends to infinity and give a numerical solution for the finite liquidation time.

In the simulation part, we give the optimal liquidation paths of the portfolio composed of two assets and conduct the sensitivity analysis of the parameters in the model. The simulation results are as follows. (1) With the increase of the threshold, the liquidation proportion of the position in the first period continuously increases, which is due to the sensitivity of the threshold to the price impact. The higher the threshold is, the more that investors ignore the price impact, which indicates that investors prefer complete liquidation in a very short time at the beginning of liquidation. (2) The volatility and loss probability are positively correlated. With the increase in volatility, investor's strategy should tend to approach simple strategies to balance the liquidation shares in different periods. In addition, it is found that the impact of the stock volatility on the loss probability is more significant than that of the related volatility component.

The conclusions of this paper have some limitations. First, the assumptions of the linear impact functions are different from the actual trading environment. Second, the asymptotic solution and finite time solution of the optimal trading strategy are given under the assumptions that the price impacts and variances of all assets are linearly related in this paper, which are too strict.

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